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Portfolio Optimization and Matrix Calculation in Excel

Description
User guide on portfolio optimization
in Excel using Excel matrix functions - 3 methods.

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1 Introduction

The purpose of this note is primarily, to make the reader able to do portfolio optimization in Excel. This note describes a series of methods for determining the optimal combination of assets in a portfolio, as well as deriving the capital market line in practice and combining portfolio optimization with investor preferences. One can consider this as guidelines on how to implement Markowitz modern portfolio theory in Excel.

The manual introduces the basics of matrix calculations in Excel in chapter 2 explaining how to use the specific matrix functions in Excel. In addition to this a brief description is provided on how matrix functions in Excel can be used to solve linear equation systems. Knowledge of matrix functions and of general matrix calculations is a prerequisite for full understanding and being able to implement portfolio optimization in Excel.

The portfolio optimization process is described in chapter 3. First, methods for calculating the return of assets and covariance matrices are outlined. These calculations are essential and form the foundation for the subsequent analysis. The portfolio optimization is then presented in a series of different calculation methods: The Black's method, the Lagrange's method and the solver method, of which Black's method is most common. A method for including a short sale restriction is introduced with the solver method.

The portfolio optimization chapter builds on a recurring example that is used throughout the chapter. The spreadsheets that are used can be found on the IKT department's manuals pages for Excel 2003 and Excel 2007: http://asb.dk/studinfo/ikt/itvaerktoejer/manualer.aspx or directly at: http://aln.hha.dk/ita/manualer/excel/EXCEL_MATRICEREGNING_OG_PORTEFOELJEOPTIMERING.pdf. This material is accessible by all students and employees.

Comments or improvement suggestions to this note are very welcome. Please, do not hesitate to e-mail analyse@asb.dk or visit the IKT Servicedesk in H14.

2 Basic matrix calculations

The knowledge and understanding of matrix calculations is a necessity for the reader to benefit from the sections about portfolio optimization. The purpose of this section is to prepare the reader to the practical work in Excel. The reader should already be familiar with fundamental matrix algebra.

Initially, Excel's matrix functions are outlined function by function. Subsequently, a description follows on how Excel matrix calculations can be applied to solve linear equation systems, since this technique is applied as a part of the portfolio optimization calculations.

The following chapter can be read and used independent of the portfolio optimization sections, as it is a general description of matrix calculation in Excel. Matrices are denoted using bold, capital letters.

2.1 Matrix functions

Excel feature four built-in matrix functions. These functions are array-type functions in the sense that they perform operations with cell ranges as input. A prerequisite for applying the functions is knowing dimensions of the output matrix. Further elaboration follows later in this chapter.

The four matrix functions:

MDETERM — Returns the determinant of a matrix. If the matrix is not quadratic, the function will return the error code "#VALUE". The determinant is used e.g. when solving equation systems using Cramer's method and to determine the rank of a matrix. Notation: $|\boldsymbol{A}|$.

Example: *=MDETERM(A1:B2)*

MINVERSE: — Returns the inverse of a matrix. If the matrix is not quadratic, the function will return the error code "#VALUE". If the matrix cannot be inverted (determinant = 0), error code "#NUM" is returned. The inverse matrix function is used when solving linear equation systems, incl. OLS optimization. Notation: $\boldsymbol{A}^{-1}$.

Example: *=MINVERSE(A1:B2)*

MMULT	Multiplies two matrixes. If dimensions do not match error code “#VALUE” is returned. To match, matrix ***A*** must have the same number of columns as the matrix ***B*** has rows – more on this in later examples. Notation: ***AB***. Example: *=MMULT(A1:B2;E1:F2)*
TRANSPOSE	Transposes the matrix. If the transposition is unsuccessful the error code “#VALUE” is returned for matrices but no errors are returned for vectors.. Notation: ***A´***. Example: *=TRANSPOSE(A1:B2)*

In Danish versions of Excel the functions are named: MDETERM, MINVERT, MPRODUKT and TRANSPONER.

When the output of the matrix operation is a matrix, only the first element of the resulting matrix is outputted in the cell. Therefore the latter 3 of the 4 functions only return the first element of the output matrix. The whole output matrix can be computed by following the instructions given below:

1. Place the cursor in the result cell and select the entire output range. One must know the dimensions of the output matrix to perform this operation and the correct dimensions must be selected.
2. Press F2.
3. Hold down CTRL+SHIFT and press ENTER. Your output matrix is filled with the lacking elements.

If you need to change an existing array calculation the procedure is as follows:

1. Select the range containing the calculated matrix/vector (if the edit results in changing dimensions, select the range corresponding to the new output matrix) and activate the function editor, e.g. by pressing F2.
2. Edit the function.
3. Leave the function editor by holding down CTRL+SHIFT and pressing ENTER. The matrix is now updated.

If the above procedure is not followed, Excel will return an error message.

2.1.1 Examples on application of Excel matrix functions

The object is to calculate the product of the two matrices ***A*** and ***B*** in Figure 2.1 below. I.e. a 2x2 matrix is multiplied by a 2x3 which returns a 2x3 matrix. The formula for this operation is:

E8 = MMULT(B3:C4;E3:G4).

Besides 13 in the result cell (E8), the rest of the matrix was produced following the guidelines above.

Figure 2.1: Multiplication

E7 | fx {=MMULT(B3:C4;E3:G4)}

	A	B	C	D	E	F	G	H
1								
2		***A***			***B***			
3		3	4		3	7	2	
4		6	1		1	2	3	
5								
6					***AB***			
7					13	29	18	
8					19	44	15	
9								

Instead, if one wants to calculate the product of ***A*** and the inverse of ***A*** ($\boldsymbol{A}^{-1}$), the following formula must be inserted in E8:

E8 = I = MMULT(MINVERSE(B3:C4);B3:C4).

Note two things about this operation. First, matrix functions can be combined and inserted into each other and second, the formula above returns the identity matrix (***I***). The calculation is not shown in the figure, because it follows from the same principle as the regular multiplication.

There are no specific matrix functions designed to add/subtract matrices in Excel. The usual operators are used: +/- signs. Now, to calculate the sum of matrix ***A*** and ***I*** (***A*** + ***I***), the following formula can be applied as shown in Figure 2.2 below.

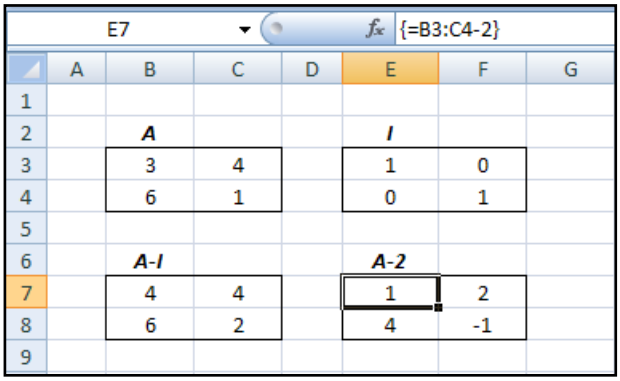
B7 = B3:C4+E3:F4.

You can also subtract a number from all the elements of a matrix by typing:

E7 = B3:C4 - 2.

Remember to conclude the operation with CTRL+SHIFT+ENTER!

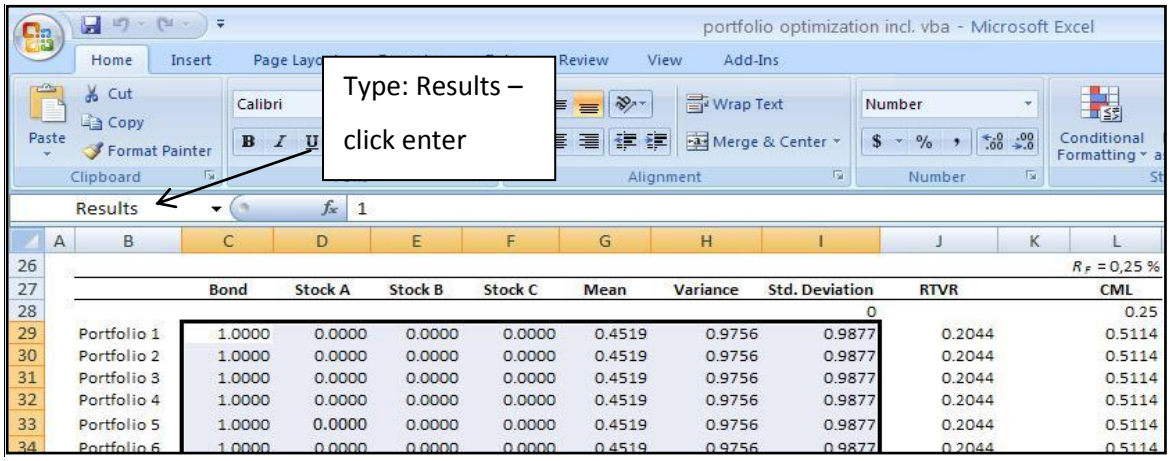
Figure 2.2: Addition and subtraction



It is often very good idea to define a name for the cell range comprising the matrix, because this makes typing and editing much more convenient. Writing formulas using range/matrix names rather than actual cell references also provides a much more intuitive and easy-to-read overview of the formulas. Naming a matrix can be done as follows (see Figure 2.3):

1. Select the cell range of the matrix
2. In the box containing the cell index in the top left corner of the window (left of the function editor), type an appropriate name for the matrix and complete the action by pressing return. This option can also be found under the tab *Formulas* using the *Define name* button. Now this name can be used as a matrix reference.

Figure 2.3: Defining a name for a range



If the matrices in Figure 2.2 had been named, the equation *B7 = B3:C4+E3:F4* could have been calculated using:

B7 = A + I.

2.2 Solving linear equation systems

Matrix calculation in Excel can provide a significant advantage when solving complex linear equation systems with many unknown variables. The section shows how to solve the general problem of *n* equations with *n* unknown variables, illustrated with an example in section 2.2.1. The principle of the below example is applied several times in the portfolio optimization section.

2.2.1 Example: 3 equations with 3 unknowns

The following linear equations are given:

$$X_1 + 3X_2 + 2X_3 = 1$$

$$3X_1 + 2X_2 + X_3 = 2$$

$$3X_1 + 4X_2 + 3X_3 = 3$$

In matrix notation the equation system can be written as:

$$\boldsymbol{AX} + \boldsymbol{B}$$

Equivalent to:

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Thus, there is *one* solution to this equations system if the matrix ***A*** is invertible. Solving for ***X*** yields:

$$\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$$

The system can be solved in Excel as shown in Figure 2.4.

Figure 2.4: Solving a linear equation system

I3 | {=MMULT(MINVERSE(A);b)}

	A	B	C	D	E	F	G	H	I	J
1										
2		***A***				***b***			Output ***x***	
3		1	3	2		1		X_1	0.75	
4		3	2	1		2		X_2	-0.75	
5		3	4	3		3		X_3	1.25	
6										

The formula in I3 is: *I3 = MMULT(MINVERSE(A);b)*

- The cell range of matrix ***A*** has been named *A* and the cell range of vector ***B*** has been named *B*.

This concludes the introduction to matrix calculations in Excel and should provide sufficient background to enable the reader to understand and implement the following instructions on how to do portfolio optimization in Excel.

3 Portfolio Optimization

Excel contains a number of functions that makes it relatively easy to model portfolio optimization. This chapter describes step-by-step how portfolio optimization is implemented using Excel. This section applies the framework of Markowitz (1952) known as Modern Portfolio Theory. Accordingly, the optimization is carried out via a trade-off between risk (standard deviation) and return taking a risk-less asset into account, thus providing the capital market line. For more on the origins of modern portfolio theory, see Tobin (1958) (introducing a risk-free asset), Lintner (1965) (CAPM), Mossin (1966) (CAPM) and Sharpe (1964) (CAPM).

The purpose of portfolio optimization is to minimize risk (volatility) for a given return (or maximize return for a given level of risk) by constructing a portfolio, that satisfies this criterion. In practice, this is carried out in two steps by 1) tracing out the efficient frontier using a given set of risky assets and 2) deriving the capital market line (CML) that is tangent to the risky-asset frontier and has intersect in the risk-free rate.

In the following sections we consider an example on real market data from three stocks and one bond. The example and preliminary calculations are presented in section 3.1 Notations and definitions are introduced in section 3.2 and three different methods for carrying out portfolio optimization in Excel are described in section 3.3 to 3.6. Lastly the optimization is combined with investor risk preferences (indifference curves) in chapter 4.

3.1 Example and Preliminary Calculations

In this section the data set that is the basis for the general example and the preliminary calculations is briefly described. The data set consists of three stocks and one bond and reaches over a period from 10 April 2003 to 10 April 2008. The stocks are DSV, Københavns Lufthavne and William Demant, and the bond is a Citigroup Weighted Government bond index, but for simplicity the assets are denoted ‘Bond’ and ‘Stock A-C’ in this note. The data is extracted in monthly frequency from Datastream. The curious reader can try re-extracting the data for completeness.

Figure 3.1: Historical market data

Market data

Date	DK CITIGROUP WGBI Bond	DSV Stock A	Kobenhavns Lufthavne Stock B	William Demant Stock C
30-04-2003	272.92	13.5	425.5	162.5
30-05-2003	277.36	14.3	466	154
30-06-2003	297.65	17.2	550	151
31-07-2003	302.62	17.3	577	149
29-08-2003	299.99	18.7	538	162.5
30-09-2003	259.66	21.1	610	186.5
31-10-2003	267.63	21.5	605	190
28-11-2003	265.2	22.95	683	207
31-12-2003	251.59	23.6	710	200.5
30-01-2004	256.06	26.8	735	198
27-02-2004	259.45	26.65	785	207.5

Source: Datastream

A set of only 4 assets does not offer much diversification potential and thus the optimization will not be optimal in the sense of application. But 4 assets are sufficient for illustrating the principles and the general idea as one can easily extend the model by adding more assets. One could consider the stocks as being indices instead, thereby combining these 3 index portfolios and reaching higher diversification. Here we denote them as stocks, though.

3.1.1 Calculating Returns

The historical stock prices from Figure 3.1 are used to calculate monthly continuously compounded returns (log-returns). Cf. Benninga (2000), this is done using the following equation:

$$R_{i,t} = \ln\left(\frac{P_{i,t}}{P_{i,t-1}}\right), \qquad \text{(Continuously compounded returns)} \qquad (3.1)$$

where $R_{i,t}$ is the return on asset *i* at time *t* and $P_{i,t}$ is price of asset *i* at time *t*. Alternatively one can use the "traditional" discrete returns:

$$R_{i,t} = \frac{P_{i,t}-P_{i,t-1}}{P_{i,t-1}}. \qquad \text{(Discrete returns)} \qquad (3.2)$$

Taking dividend payments into account these return formulas can be written as:

$$R_{i,t} = \ln\left(\frac{P_{i,t}+D_{i,t}}{P_{i,t-1}}\right); \qquad \text{(Continuously compounded returns)} \qquad (3.3)$$

$$R_{i,t} = \frac{P_{i,t} - P_{i,t-1} + D_{i,t}}{P_{i,t-1}}, \qquad \text{(Discrete returns)} \qquad (3.4)$$

where $D_{i,t}$ is the dividend payment on asset *i* at time *t*. In finance most often the continuously compounded returns are used. Since the assets in the data set does not pay out any dividends in the period (or are already adjusted for dividends) formula (3.1) is applied. This yields:

Figure 3.2: Calculated returns

C6 =LN('Market data'!C7/'Market data'!C6)*100

Returns

Date	Bond	Stock A	Stock B	Stock C
30-04-2003				
30-05-2003	1.6138	5.7570	9.0921	-5.3725
30-06-2003	7.0602	18.4650	16.5733	-1.9673
31-07-2003	1.6560	0.5797	4.7924	-1.3334
29-08-2003	-0.8729	7.7817	-6.9984	8.6732
30-09-2003	-14.4376	12.0750	12.5600	13.7753
31-10-2003	3.0232	1.8780	-0.8230	1.8593
28-11-2003	-0.9121	6.5265	12.1266	8.5695
31-12-2003	-5.2683	2.7929	3.8770	-3.1905
30-01-2004	1.7611	12.7155	3.4606	-1.2547
27-02-2004	1.3152	-0.5613	6.5813	4.6864
31-03-2004	1.6930	-2.2771	-1.9293	-0.9685
30-04-2004	5.9688	1.3346	6.7780	9.7299
31-05-2004	-0.6267	0.7547	2.2796	-5.2107
30-06-2004	0.0359	9.3257	5.7620	7.3942
30-07-2004	6.2031	0.1711	-4.1149	1.9252
31-08-2004	2.9171	2.5318	2.9887	-11.0790
30-09-2004	2.5568	2.9559	4.1048	16.2401
29-10-2004	1.5284	3.9656	-1.8651	5.2902

As can be seen from Figure 3.2 one observation is lost because the first return calculation requires the first two price observations, thereby using the price observation from time *t – 1* and time *t*.

3.1.2 Calculating the Covariance Matrix

Calculating the covariance matrix is a fundamental part of the portfolio optimization. Cf. Elton et al. (2007) the covariance can be calculated using the following formula:

$$(\sigma_{ik}) = E[(R_{it} - \bar{R}_i)(R_{kt} - \bar{R}_k)] = \frac{1}{T}\sum_{t=1}^{T}(R_{it} - \bar{R}_i)(R_{kt} - \bar{R}_k), \qquad (3.5)$$

where $\bar{R}_i$ and $\bar{R}_k$ are mean returns for asset *i* and *j*, respectively, and R_{it} and R_{kt} are return at time *t* on asset *i* and *k*, respectively. The sum of multiplied deviations in each asset is multiplied by $\frac{1}{T}$ indicating that the observations are seen as a population. Using $\frac{1}{T-1}$ alternatively, one takes into account the loss of degrees of freedom associated with using a sample rather than the population (estimating the variance

parameter rather than knowing it). The latter is specifically important for small samples, but in this example $\frac{1}{T}$ is used for the sake of simplicity and because the sample is fairly large. Note that the Excel covariance function (=COVAR()) also uses $\frac{1}{T}$, while the Excel variance function (=VAR()) uses $\frac{1}{T-1}$.

Before calculating the covariance matrix, the mean returns for each asset are needed to calculate the return deviations from mean. This is done using the following expression:

$$\bar{R}_i = \frac{1}{T}\sum_{t=1}^{T} R_{it}, \qquad (3.6)$$

where $\bar{R}_i$ is the mean return for asset *i*, which is calculated as the sum of returns divided by number of observation. The calculation of mean returns for the 4 assets is illustrated in Figure 3.3.

Note that also the Excel function =AVG() (in Danish: =MIDDEL()) can be used to calculate the mean.

Figure 3.3: Calculation of mean returns

C66 | =AVERAGE(C6:C65)

	A	B	C	D	E	F	G
55		29-06-2007	-0.8995	-4.7731	4.0039	2.9146	
56		31-07-2007	-0.5120	4.5611	-0.3276	-2.9146	
57		31-08-2007	2.2692	8.3467	2.5113	-0.1850	
58		28-09-2007	1.4467	-4.1880	0.5982	-19.0130	
59		31-10-2007	0.2755	8.7690	-0.9990	3.0873	
60		30-11-2007	1.6603	-6.1558	-1.6195	-4.5538	
61		31-12-2007	-2.8627	-3.8427	-2.9408	8.8053	
62		31-01-2008	-1.1587	-20.7798	-2.9433	-14.4142	
63		29-02-2008	3.6101	-4.6762	0.1730	-24.6860	
64		31-03-2008	1.1642	1.0582	-2.4055	8.2643	
65		30-04-2008	-0.9630	10.2462	-2.6465	7.2387	
66		Average	-0.4059	3.4227	2.7382	1.4136	
67							

The covariances can now be calculated using formula (3.5) but this is relatively inconvenient especially if the number of assets is increased. Instead matrix notation and calculation is applied. To calculate the covariance matrix using matrix calculation functions requires the excess returns one will have to calculate the return deviations from mean. That is, the excess returns, which is defined by:

$$Excess\ return = R_{it} - \bar{R}_i, \qquad (3.7)$$

where R_{it} is return on asset *i* at time *t* and $\bar{R}_i$ is the mean return on asset *i* (Elton et al. (2007)). Note that the excess return is part of formula (3.5). In total the excess returns can be written in matrix form as:

$$Excess\ return\ (R_{it} - \bar{R}_i) = \begin{bmatrix} R_{11} - \bar{R}_1 & R_{21} - \bar{R}_2 & \cdots & R_{I1} - \bar{R}_I \\ R_{12} - \bar{R}_1 & R_{22} - \bar{R}_2 & \cdots & R_{I1} - \bar{R}_I \\ \vdots & \vdots & \ddots & \vdots \\ R_{1N} - \bar{R}_1 & R_{2N} - \bar{R}_2 & \cdots & R_{IN} - \bar{R}_I \end{bmatrix}$$

This matrix can be calculated in Excel as is illustrated in

Figure 3.4.

Figure 3.4: Calculation of excess returns matrix

C6 | =Returns!C6-Returns!C$66

Excess Returns

Date	Bond	Stock A	Stock B	Stock C
30-04-2003				
30-05-2003	2.020	2.334	6.354	-6.786
30-06-2003	7.466	15.042	13.835	-3.381
31-07-2003	2.062	-2.843	2.054	-2.747
29-08-2003	-0.467	4.359	-9.737	7.260
30-09-2003	-14.032	8.652	9.822	12.362
31-10-2003	3.429	-1.545	-3.561	0.446
28-11-2003	-0.506	3.104	9.388	7.156
31-12-2003	-4.862	-0.630	1.139	-4.604
30-01-2004	2.167	9.293	0.722	-2.668
27-02-2004	1.721	-3.984	3.843	3.273

When excess return matrix $(R_{it} - \bar{R}_i)$ is calculated the covariance matrix can be constructed. The covariance matrix is given by:

$$(\sigma_{ik}) = \frac{(\boldsymbol{R_{it}} - \bar{\boldsymbol{R}}_i)'(\boldsymbol{R_{it}} - \bar{\boldsymbol{R}}_i)}{T}, \qquad \text{Covariance matrix (population)} \qquad (3.8)$$

where $(\boldsymbol{R_{it}} - \bar{\boldsymbol{R}}_i)$ is the excess return matrix, $(\boldsymbol{R_{it}} - \bar{\boldsymbol{R}}_i)'$ is the transposed excess return matrix and T is the number of observations or time periods. In smaller samples the corresponding covariance formula is as follows, cf. above on loss of degrees of freedom:

$$(\sigma_{ik}) = \frac{(\boldsymbol{R_{it}} - \bar{\boldsymbol{R}}_i)'(\boldsymbol{R_{it}} - \bar{\boldsymbol{R}}_i)}{T-1}, \qquad \text{Covariance matrix (sample)} \qquad (3.9)$$

Applying the Excel matrix functions from section 2, the covariance matrix for the population (3.8) can be calculated applying the below expression:

$$(\sigma_{ik}) = MMULT(TRANSPOSE(XR); XR)/T. \tag{3.10}$$

See Figure 3.5 for the calculation.

Figure 3.5: Calculation of the covariance matrix in Excel

C5 {=MMULT(TRANSPOSE(XR);XR)/(COUNT(Returns!C6:C65))}

Variance/Covariance Matrix

	Bond	Stock A	Stock B	Stock C
Bond	44.14	5.74	-0.85	-1.62
Stock A	5.74	68.34	4.04	26.84
Stock B	-0.85	4.04	34.10	4.63
Stock C	-1.62	26.84	4.63	54.30

Correlation Matrix

	Bond	Stock A	Stock B	Stock C
Bond	1.00	0.10	-0.02	-0.03
Stock A	0.10	1.00	0.08	0.44
Stock B	-0.02	0.08	1.00	0.11
Stock C	-0.03	0.44	0.11	1.00

With the preliminary calculations carried out one can now proceed with the actual portfolio optimization techniques.

3.2 Notations and Definitions

The purpose of this section is to clarify notations and definitions through a general introduction, thus having established the framework to be used throughout the rest of this note. The notation in this note is consistent with the notation used in the course Asset Pricing I (Stefan Hirth) and to some extend Elton et al (2007).

To find the optimal portfolio the objective is to compute the optimal portfolio weights. These portfolio weights are defined by:

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}, \tag{3.11}$$

where $\boldsymbol{X}$ is the vector containing the portfolio weights and X_i is the portfolio weight assigned to asset i. The number of assets is denoted N. Thus, the transpose of the weight vector ($\boldsymbol{X}'$) can be defined as:

$$\boldsymbol{X}' = [X_1 \quad X_2 \quad \dots \quad X_N] \tag{3.12}$$

Likewise the expected returns on the assets need to be computed. That is, the mean returns. On vector form this can be written as:

$$E[\boldsymbol{R}] = \overline{\boldsymbol{R}} = \begin{bmatrix} E[R_1] \\ E[R_2] \\ \vdots \\ E[R_N] \end{bmatrix} = \begin{bmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \vdots \\ \bar{R}_N \end{bmatrix}, \qquad \text{(expected return)} \tag{3.13}$$

where $E[\boldsymbol{R}]$ denotes the vector containing the expected returns and $E[\boldsymbol{R}_i]$ denotes the expected return on asset i. Combining the weights of the assets with the expected returns on each asset, the total portfolio return can be found by computing the weighted average:

$$E[R_P] = \bar{R}_P = \boldsymbol{X}'\overline{\boldsymbol{R}} = \sum_{i=1}^{N} X_i \bar{R}_i \qquad \text{(portfolio return)} \tag{3.14}$$

The variance on the portfolio can be calculated as:

$$Var(R_P) = \boldsymbol{X}' \cdot (\sigma_{ik}) \cdot \boldsymbol{X} = \sum_{i=1}^{N} ({X_i}^2 \sigma_i^2) + \sum_{i=1}^{N} \sum_{\substack{k=1 \\ k \neq i}}^{N} (X_i X_k \sigma_{ik}) = \sum_{i=1}^{N} \sum_{k=1}^{N} (X_i X_k \sigma_{ik})$$

$$\text{(portfolio variance)} \tag{3.15}$$

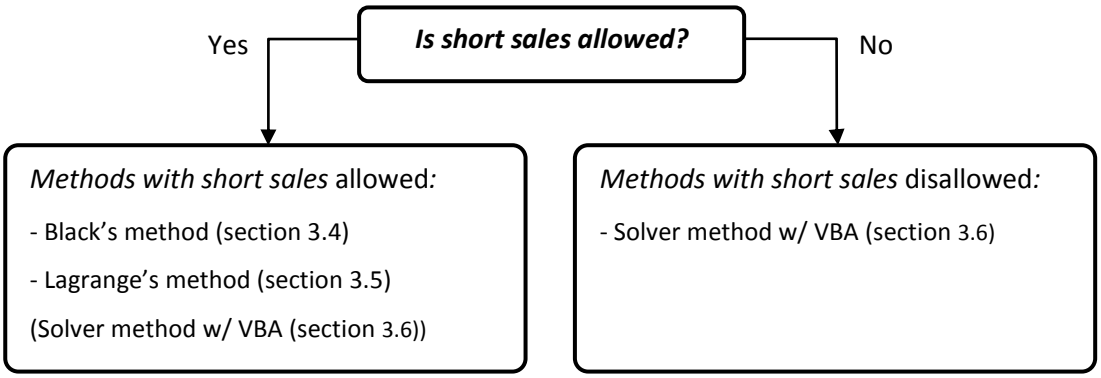
where (σ_{ik}) is the covariance matrix calculated in section 3.1.2 (Elton et al. (2007)).

3.3 Methods of Portfolio Optimization Excel

As mentioned in the introduction to this chapter the purpose of portfolio optimization is to find the so called efficient frontier that is traced out by the portfolios that minimize risk of a given return. This is a complex matter, and therefore several methods for computing the efficient frontier in practice exist. In this section three methods are presented, cf. Figure 3.6.

As can be seen from Figure 3.6, in this note, the methods are divided on whether or not short sale is allowed. For this note, little focus will be on the short sale restriction as it easily implemented/disregarded in the solver solution, while implementation in the Black's and Lagrange's method requires great and complicated extensions that goes beyond the scope of this note. The primary focus will be on Black's method.

Figure 3.6: Methodology



The reason for introducing three methods of optimization is the method's different level of complexity. The solver method is the most intuitive and as such also the method that has least requirements to the reader's knowledge of matrix algebra and econometric techniques. Unfortunately the solver method also is the most time consuming method and provides the least accurate results. So, one must trade-off accuracy for calculation complexity. Both Black's method and especially Lagrange's method require thorough knowledge and understanding of matrix algebra and econometrics. These methods are more accurate and automatic methods of determining the efficient frontier, though. The reader's knowledge of matrix calculations in Excel should determine the choice, keeping in mind that Black's method is the most relevant for the courses taught at Aarhus School of Business. For combining portfolio optimization with investor risk preferences in the form of utility maximization see chapter 0.

3.4 Black's method

This section provides a practical guide to the implementation and use of Black's method to determine the optimal portfolio without short sales restriction. The portfolio optimization problem can be constructed as a minimization problem (minimizing portfolio variance) with an object function and a set of constraints (Elton et al (2007)):

Object function: $$\min_{(X_1,X_2,\ldots,X_N)} \ \sigma_P^2 = \boldsymbol{X}' \cdot (\sigma_{ik}) \cdot \boldsymbol{X} \qquad (3.16)$$

Constraints: $\sum X_i E[R_i] = E[R_P]$ (Portfolio return) (3.17)

$\sum X_i = 1$ (The sum of portfolio weights equals 1) (3.18)

The expression states that portfolio variance is minimized by adjusting portfolio weights, X_i. The expression is conditioned on the relation between portfolio return and the sum of weighted individual returns (3.17) and the portfolio weights summing to 1 (3.18).

Then the Black's optimization method is as follows:

- Estimate 2 efficient portfolios (X and Y) by Black's algorithm (3.4.1).
- Estimate the efficient frontier (all efficient portfolios) by combining portfolio X and Y (3.4.2).
- Derive the capital market line (CML) (3.4.3).

3.4.1 Estimating the X and Y portfolios

Cf. Benninga (2000) Black's estimation of efficient portfolios takes the below linear equation system as starting point:

$$E[\boldsymbol{R}] - c = (\sigma_{ik}) \cdot \boldsymbol{Z}, \tag{3.19}$$

Where *[E(**R**) – c]* is the vector of expected returns subtracted by a constant *c* (scalar), ***Ω*** is the covariance matrix and ***Z*** is some vector that solves the equation. Solving for ***Z*** yields:

$$\boldsymbol{Z} = (\sigma_{ik})^{-1}\,[E(\boldsymbol{R}) - c] \quad \text{or equivalently} \quad \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_N \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN} \end{bmatrix}^{-1} \begin{bmatrix} E(R_1) - c \\ E(R_2) - c \\ \vdots \\ E(R_N) - c \end{bmatrix}, \tag{3.20}$$

The vector ***Z*** can be considered the unstandardized asset weights in the portfolio. Accordingly, the standardized and more meaningful weights X_i can be computed in the following way:

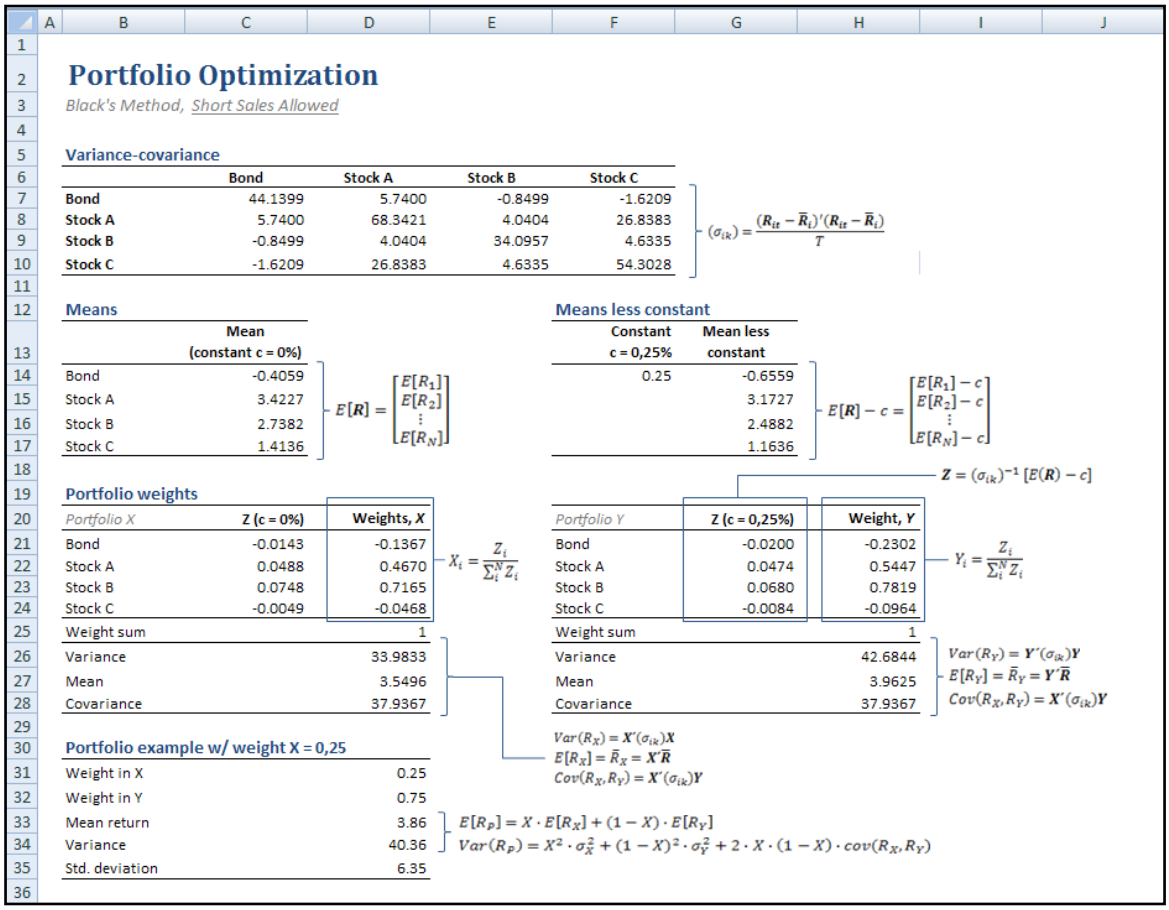
$$X_i = \frac{Z_i}{\sum_i^N Z_i}, \quad \text{alternatively} \quad Y_i = \frac{Z_i}{\sum_i^N Z_i} \text{ for portfolio Y,} \tag{3.21}$$

where X_i are the weights in some arbitrary efficient portfolio. The corresponding expected portfolio returns $E[R_P]$ and variances $Var(R_P)$ can be calculated using (3.14) and (3.15) (Benninga (2000)).

Essentially, the weights in the efficient portfolios X and Y can now be calculated using (3.21), setting *c* equal to *0* and some rate (an arbitrary scalar greater than 0) that can be considered the risk-free rate (or a "quasi-risk-free rate"), respectively. *c* has some interesting interpretation. The obvious choice for *c* is the risk-free rate if a riskless asset exists. Alternatively, if no riskless asset exists, *c* has close resemblance to the zero-beta rate called R_Z (from the zero-beta CAPM). In fact *c* equals $E[R_Z]$ if the estimated market (tangent) portfolio is efficient. In this note, the case where a risk-free asset exists is considered for simplicity. In the following, *c* is used in the calculation techniques and R_F is used for interpretation purposes and economic understanding.

In Figure 3.7 below, Black's method is outlined applying it on the example introduced in section 3.1. The figure shows how the efficient portfolios *X* and *Y* can be calculated with help from the corresponding formulas.

Figure 3.7: Set-up for Black's method



Portfolio Optimization

Black's Method, Short Sales Allowed

Variance-covariance

	Bond	Stock A	Stock B	Stock C
Bond	44.1399	5.7400	-0.8499	-1.6209
Stock A	5.7400	68.3421	4.0404	26.8383
Stock B	-0.8499	4.0404	34.0957	4.6335
Stock C	-1.6209	26.8383	4.6335	54.3028

$$(\sigma_{ik}) = \frac{(R_{it} - \bar{R}_i)'(R_{it} - \bar{R}_i)}{T}$$

Means

	Mean (constant c = 0%)
Bond	-0.4059
Stock A	3.4227
Stock B	2.7382
Stock C	1.4136

$$E[\boldsymbol{R}] = \begin{bmatrix} E[R_1] \\ E[R_2] \\ \vdots \\ E[R_N] \end{bmatrix}$$

Means less constant

Constant c = 0,25%	Mean less constant
0.25	-0.6559
	3.1727
	2.4882
	1.1636

$$E[\boldsymbol{R}] - c = \begin{bmatrix} E[R_1] - c \\ E[R_2] - c \\ \vdots \\ E[R_N] - c \end{bmatrix}$$

$$\boldsymbol{Z} = (\sigma_{ik})^{-1}\,[E(\boldsymbol{R}) - c]$$

Portfolio weights

Portfolio X	Z (c = 0%)	Weights, X
Bond	-0.0143	-0.1367
Stock A	0.0488	0.4670
Stock B	0.0748	0.7165
Stock C	-0.0049	-0.0468
Weight sum		1
Variance		33.9833
Mean		3.5496
Covariance		37.9367

$$X_i = \frac{Z_i}{\sum_i^N Z_i}$$

Portfolio Y	Z (c = 0,25%)	Weight, Y
Bond	-0.0200	-0.2302
Stock A	0.0474	0.5447
Stock B	0.0680	0.7819
Stock C	-0.0084	-0.0964
Weight sum		1
Variance		42.6844
Mean		3.9625
Covariance		37.9367

$$Y_i = \frac{Z_i}{\sum_i^N Z_i}$$

$$Var(R_Y) = \boldsymbol{Y}'(\sigma_{ik})\boldsymbol{Y}$$
$$E[R_Y] = \bar{R}_Y = \boldsymbol{Y}'\bar{\boldsymbol{R}}$$
$$Cov(R_X, R_Y) = \boldsymbol{X}'(\sigma_{ik})\boldsymbol{Y}$$

$$Var(R_X) = \boldsymbol{X}'(\sigma_{ik})\boldsymbol{X}$$
$$E[R_X] = \bar{R}_X = \boldsymbol{X}'\bar{\boldsymbol{R}}$$
$$Cov(R_X, R_Y) = \boldsymbol{X}'(\sigma_{ik})\boldsymbol{Y}$$

Portfolio example w/ weight X = 0,25

Weight in X	0.25
Weight in Y	0.75
Mean return	3.86
Variance	40.36
Std. deviation	6.35

$$E[R_P] = X \cdot E[R_X] + (1 - X) \cdot E[R_Y]$$
$$Var(R_P) = X^2 \cdot \sigma_X^2 + (1 - X)^2 \cdot \sigma_Y^2 + 2 \cdot X \cdot (1 - X) \cdot cov(R_X, R_Y)$$

Portfolio *X*, i.e. when *c* = 0, is then calculated step-by-step. The vector ***Z*** is calculated by inserting the covariance matrix and the mean returns, respectively, in (3.20):

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} = \begin{bmatrix} 44.1399 & 5.7400 & -0.8499 & -1.6209 \\ 5.7400 & 68.3421 & 4.0404 & 26.8383 \\ -0.8499 & 4.0404 & 34.0957 & 4.6335 \\ -1.6209 & 26.8383 & 4.6335 & 54.3028 \end{bmatrix}^{-1} \begin{bmatrix} -0.4059 - 0 \\ 3.4227 - 0 \\ 2.7382 - 0 \\ 1.4136 - 0 \end{bmatrix} = \begin{bmatrix} -0.0143 \\ 0.0488 \\ 0.0748 \\ -0.0049 \end{bmatrix}$$

The ***Z*** coordinates are then inserted in formula (3.21), which yields the ***X*** vector of portfolio weights:

$$\boldsymbol{X} = \begin{bmatrix} Bond \\ Stock\ A \\ Stock\ B \\ Stock\ C \end{bmatrix} = \begin{bmatrix} \dfrac{-0.0143}{-0.0143 + 0.0488 + 0.0748 - 0.0049} \\ \dfrac{0.0488}{-0.0143 + 0.0488 + 0.0748 - 0.0049} \\ \dfrac{0.0748}{-0.0143 + 0.0488 + 0.0748 - 0.0049} \\ \dfrac{-0.0049}{-0.0143 + 0.0488 + 0.0748 - 0.0049} \end{bmatrix} = \begin{bmatrix} -0.1367 \\ 0.4670 \\ 0.7165 \\ -0.0468 \end{bmatrix}$$

Thereby, the portfolio weights for portfolio *X* are outputted and the portfolio's expected return and variance can be calculated using (3.13) and (3.16):

$$E[R_X] = \boldsymbol{X}'\overline{\boldsymbol{R}} = \begin{bmatrix} -0.1367 \\ 0.4670 \\ 0.7165 \\ -0.0468 \end{bmatrix}' \begin{bmatrix} -0.4059 \\ 3.4227 \\ 2.7382 \\ 1.41360 \end{bmatrix} = \underline{3.5496}$$

$$Var(R_X) = \boldsymbol{X}'(\sigma_{ik})\boldsymbol{X} = \begin{bmatrix} -0.1367 \\ 0.4670 \\ 0.7165 \\ -0.0468 \end{bmatrix}' \begin{bmatrix} 44.1399 & 5.7400 & -0.8499 & -1.6209 \\ 5.7400 & 68.3421 & 4.0404 & 26.8383 \\ -0.8499 & 4.0404 & 34.0957 & 4.6335 \\ -1.6209 & 26.8383 & 4.6335 & 54.3028 \end{bmatrix} \begin{bmatrix} -0.1367 \\ 0.4670 \\ 0.7165 \\ -0.0468 \end{bmatrix} = \underline{33.9833}$$

After computing the efficient portfolio *X*, the same procedure can be followed to calculate the efficient portfolio *Y* setting *c* equal to a constant resembling the risk-free rate. In this example *c* is set equal to 0.25% (per month). These calculations are not shown here, since they are simply a replication of the computations for *X*. Note, though, that the estimated *Y* portfolio will be equal to the so-called market portfolio when *c* is set equal to the risk-free rate R_F.

3.4.2 Estimating the efficient frontier

In modern portfolio theory it is of particular interest to trace out the critical frontier, since this consists of all efficient portfolios given the available assets. It represents the set entire set of portfolios that are perfectly diversified – that is, all portfolios (combinations of assets) that yield maximum return for a given level of risk/std. dev. Black (1972) has shown that the entire efficient frontier can be traced out by combinations of just 2 efficient portfolios – in this case *X* and *Y*.

The principle is that given two efficient portfolios, all efficient portfolios will be combinations of these two. Given an arbitrary constant *X* (the weight in portfolio *X*) and 2 efficient portfolios *X* and *Y*, this means, that the following equation will produce an efficient portfolio, cf. Benninga (2000):

$$w \cdot \boldsymbol{X} + (1-w) \cdot \boldsymbol{Y} = \begin{bmatrix} w \cdot X_1 + (1-w) \cdot Y_1 \\ w \cdot X_2 + (1-w) \cdot Y_2 \\ \vdots \\ w \cdot X_N + (1-w) \cdot Y_N \end{bmatrix} \tag{3.22}$$

All efficient portfolios can now be calculated by adjusting the constant *w*. This provides a very quick and easy way of tracing out the efficient frontier. When efficient portfolios are generated by (3.22), the corresponding expected returns $E[R_p]$ and variances $Var(R_p)$ can be calculated using the following equations, respectively:

$$E[R_P] = X \cdot E[R_X] + (1-X) \cdot E[R_Y] \tag{3.23}$$

$$Var(R_P) = X^2 \cdot \sigma_X^2 + (1-X)^2 \cdot \sigma_Y^2 + 2 \cdot X \cdot (1-X) \cdot cov(X,Y) \tag{3.24}$$

Where *w* is the share of portfolio *X* in portfolio *P*. The covariance *Cov(X,Y)* can be calculated as follows:

$$Cov(X,Y) = \boldsymbol{X}'(\sigma_{ik})\boldsymbol{Y}, \tag{3.25}$$

where the vectors *X* and *Y* contain the asset weights in portfolios *X* and *Y*, respectively, and (σ_{ik}) is the covariance matrix.

The above equations and procedure are implemented in Figure 3.7. As can be seen in the figure, it is relatively easy to construct efficient portfolios by changing the constant *w*.

By restructuring the worksheet in Figure 3.7 it is very easy to create a series of efficient portfolios of which the optimal portfolio can later be selected. These somewhat auxiliary portfolios are illustrated in Figure 3.8:

Figure 3.8: Computing the efficient frontier

	A	B	C	D	E	F	G	H	I	J	K	L	M
38													
39		Auxiliary portfolios									Reward-To-Variability Ratio		R_F = 0,25 %
40			Bond	Stock A	Stock B	Stock C	Share of X	Expected return	Variance	Std.dev	RTVR (slope)		CML
41											0		0.2500
42		Portfolio 1	-0.5106	0.7780	0.9780	-0.2454	-3.0	5.20	78.3190	8.8498	0.5595		5.2789
43		Portfolio 2	-0.4639	0.7392	0.9453	-0.2206	-2.5	4.99	71.3871	8.4491	0.5616		5.0512
44		Portfolio 3	-0.4171	0.7003	0.9126	-0.1958	-2.0	4.79	64.8522	8.0531	0.5636		4.8261
45		Portfolio 4	-0.3704	0.6614	0.8799	-0.1709	-1.5	4.58	58.7146	7.6625	0.5653		4.6042
46		Portfolio 5	-0.3237	0.6225	0.8473	-0.1461	-1.0	4.38	52.9740	7.2783	0.5668		4.3859
47		Portfolio 6	-0.2769	0.5836	0.8146	-0.1213	-0.5	4.17	47.6306	6.9015	0.5679		4.1718
48		Portfolio 7	-0.2302	0.5447	0.7819	-0.0964	0.0	3.96	42.6844	6.5333	0.5682	TP	3.9625
49		Portfolio 8	-0.1834	0.5059	0.7492	-0.0716	0.5	3.76	38.1353	6.1754	0.5678		3.7591
50		Portfolio 9	-0.1367	0.4670	0.7165	-0.0468	1.0	3.55	33.9833	5.8295	0.5660		3.5626
51		Portfolio 10	-0.0899	0.4281	0.6838	-0.0220	1.5	3.34	30.2285	5.4980	0.5626		3.3743
52		Portfolio 11	-0.0432	0.3892	0.6511	0.0029	2.0	3.14	26.8708	5.1837	0.5569		3.1956
53		Portfolio 12	0.0035	0.3503	0.6184	0.0277	2.5	2.93	23.9103	4.8898	0.5481		3.0286
54		Portfolio 13	0.0503	0.3114	0.5857	0.0525	3.0	2.72	21.3469	4.6203	0.5354		2.8755
55		Portfolio 14	0.0970	0.2726	0.5530	0.0774	3.5	2.52	19.1807	4.3796	0.5177		2.7387
56		Portfolio 15	0.1438	0.2337	0.5204	0.1022	4.0	2.31	17.4116	4.1727	0.4939		2.6211
57		Portfolio 16	0.1905	0.1948	0.4877	0.1270	4.5	2.10	16.0397	4.0050	0.4630		2.5258
58		Portfolio 17	0.2373	0.1559	0.4550	0.1519	5.0	1.90	15.0649	3.8813	0.4246		2.4556
59		Portfolio 18	0.2840	0.1170	0.4223	0.1767	5.5	1.69	14.4872	3.8062	0.3787		2.4129
60		Portfolio 19	0.3307	0.0781	0.3896	0.2015	6.0	1.48	14.3067	3.7824	0.3265	GMV	2.3994
61		Portfolio 20	0.3775	0.0393	0.3569	0.2264	6.5	1.28	14.5233	3.8109	0.2699		2.4156
62		Portfolio 21	0.4242	0.0004	0.3242	0.2512	7.0	1.07	15.1371	3.8906	0.2113		2.4608
63		Portfolio 22	0.4710	-0.0385	0.2915	0.2760	7.5	0.87	16.1480	4.0185	0.1532		2.5335
64		Portfolio 23	0.5177	-0.0774	0.2588	0.3008	8.0	0.66	17.5561	4.1900	0.0976		2.6310
65		Portfolio 24	0.5644	-0.1163	0.2261	0.3257	8.5	0.45	19.3613	4.4002	0.0460		2.7504
66		Portfolio 25	0.6112	-0.1551	0.1935	0.3505	9.0	0.25	21.5637	4.6437	-0.0008		2.8888
67		***Note*** *that when inputting new data, you always need to adjust the 'Share of X' step size to fit the data. If the GMV portfolio or TP portfolio are indicated in the auxiliary portfolios to be in one of the extremes, this might be incorrect since the indicators are just simple Excel conditional formatting of minimum variance and maximum RTVR. Thus the correct GMV/TP portfolio might be located outside the range of the 25 auxiliary portfolios. This can be corrected by adjusting the step size in 'Share of X'.*											
68													

3.4.3 Deriving the capital market line

All the calculated portfolios are now reviewed for their "Reward-To-Variability Ratio". This indicator measures the risk premium per unit of risk and is calculated by the following formula (Elton et al (2007)):

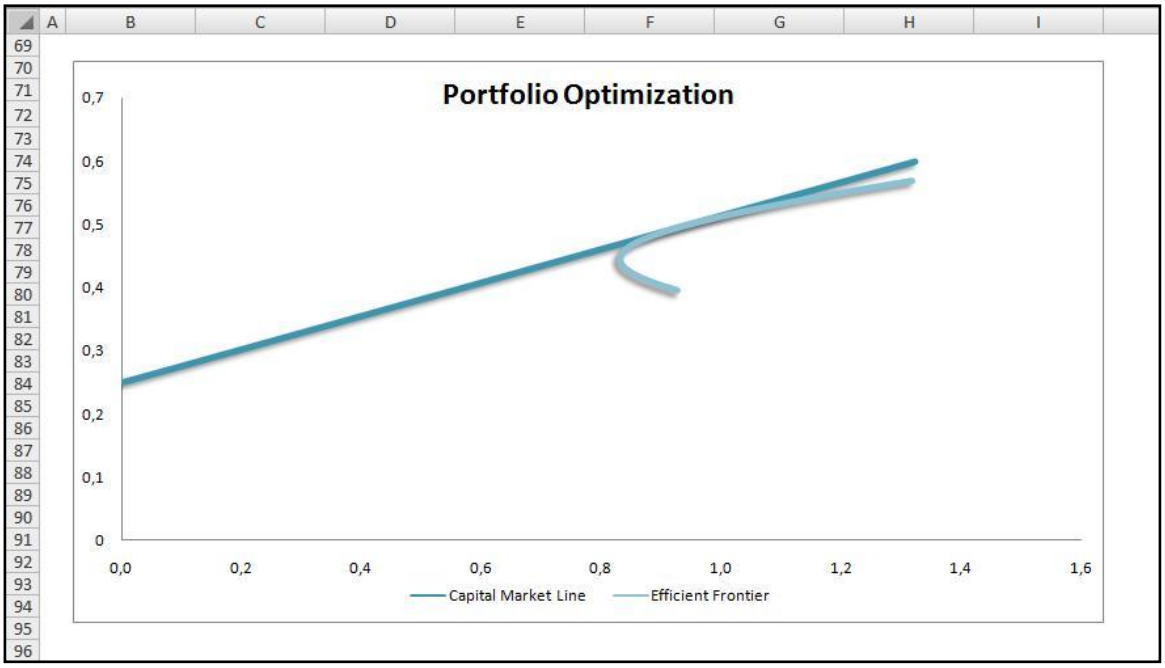
$$RTVR = \frac{E[R_P] - R_F}{\sigma_P} \tag{3.26}$$

Investors are interested in maximizing RTVR since this yields the highest possible return given the associated risk. Accordingly, the optimal portfolio can be found where RTVR is greatest, and is called the market portfolio. When a risk-free asset is introduced, one can construct a new efficient frontier consisting of combinations of the market portfolio and the risk-free asset, according to (3.22), thereby reaching a new series of optimal portfolios. Imposing a straight line between the risk-free asset and the market portfolio, yields the "capital market line". The capital market line has intersect in R_F, is tangent to the market portfolio (MP or TP for Tangent Portfolio) and has slope of RTVR.

Note that the market portfolio is found exactly where the share invested in portfolio *X* is zero. The market portfolio is therefore identical to the *Y* portfolio, when $c = R_F$ as mentioned earlier.

The efficient frontier and CML are illustrated in figure 3.9, constructed using the Excel scatter plot function with soft curves. Risk/std. dev. on the x-axis, expected return on the y-axis:

Figure 3.9: Graphical illustration of the efficient frontier and CML



3.5 Lagrange's method

The Lagrange's method is applicable to finding the optimal portfolio when short sales are allowed. The optimization problem is similar to the one used in Black's method given in (3.17).

Object function: $\min_{(X_1,X_2,\ldots,X_N)} \ \sigma_P^2 = \boldsymbol{X}'(\sigma_{ik})\boldsymbol{X}$ (3.27)

Constraints: $\sum X_i E(R_i) = E(R_P)$ (Portfolio return) (3.28)

$\sum X_i = 1$ (The sum of portfolio weights equals 1) (3.29)

The Lagrange's method applies an analytical approach to solving (3.27). Thus, the Lagrange's equation can be written as (3.30) Greene (2003):

$$L(\boldsymbol{X},\boldsymbol{\lambda}) = \boldsymbol{X}'(\sigma_{ik})\boldsymbol{X} - \boldsymbol{\lambda}'(\boldsymbol{R}'\boldsymbol{X} - \boldsymbol{\mu}) \quad (3.30)$$

In this case $\boldsymbol{X}$ represents a vector of portfolio weights. (σ_{ik}) represents the covariance matrix, $\boldsymbol{\lambda}$ is a vector which holds the Lagrange's multipliers, $\boldsymbol{R}$ is a $[i \times 2]$ matrix that contains the expected returns in the first column and ones in the second column. The vector $\boldsymbol{\mu}$ has dimensions $[2 \times 1]$ with mean portfolio return in the first row and one in the second row.

The solution to the Lagrange's equation follows from differentiating (3.30). This derivation is beyond the scope of this manual; hence the results will be applied as given. For further reference to the derivation, see Greene (2003).

Differentiating (3.30) yields

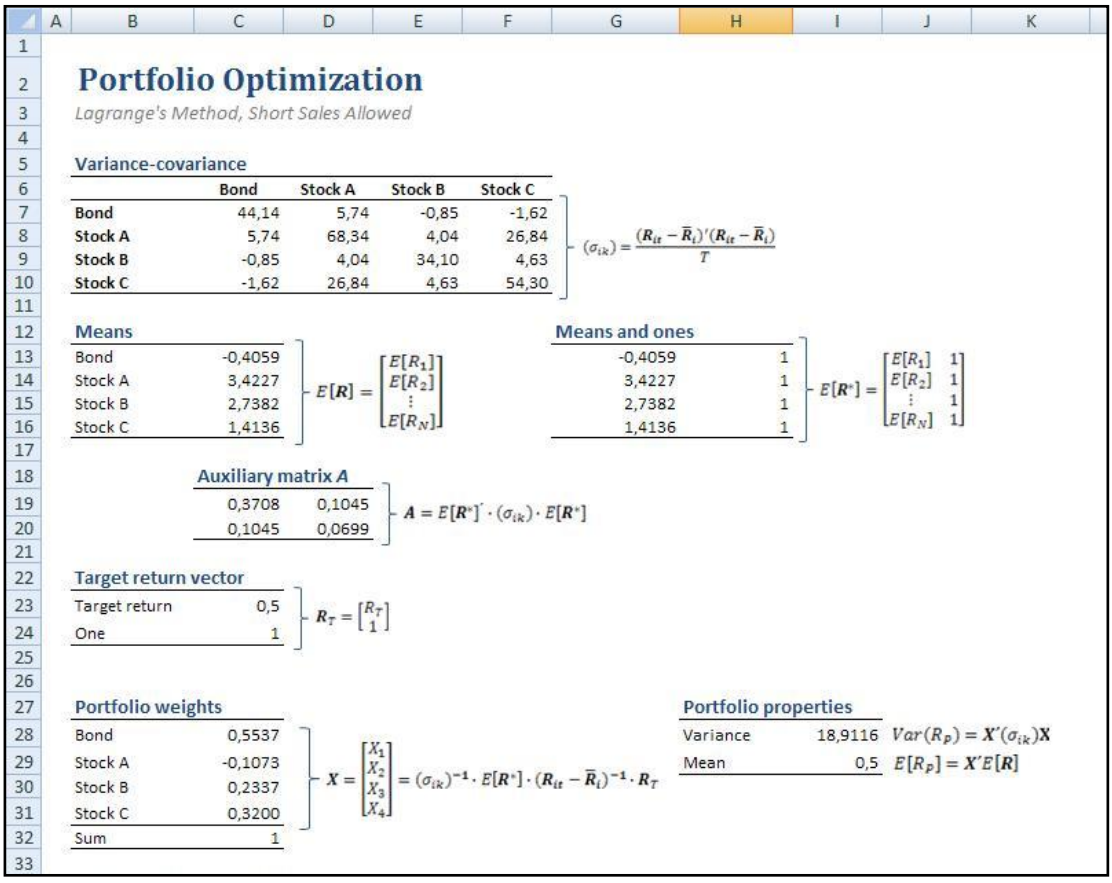
$$\boldsymbol{X} = [(\sigma_{ik})^{-1} \cdot \boldsymbol{R} \cdot \boldsymbol{A}^{-1} \cdot \boldsymbol{\mu}], \tag{3.31}$$

where the matrix $\boldsymbol{A}$ is given by the following $[2 \times 2]$ matrix:

$$\boldsymbol{A} = [\boldsymbol{R}' \, (\sigma_{ik})^{-1} \, \boldsymbol{R}] \tag{3.32}$$

The implementation in an Excel can be seen in the below Figure 3.10:

Figure 3.10: Implementation of the Lagrange's method



Portfolio Optimization

Lagrange's Method, Short Sales Allowed

Variance-covariance

	Bond	Stock A	Stock B	Stock C
Bond	44,14	5,74	-0,85	-1,62
Stock A	5,74	68,34	4,04	26,84
Stock B	-0,85	4,04	34,10	4,63
Stock C	-1,62	26,84	4,63	54,30

$$(\sigma_{ik}) = \frac{(\boldsymbol{R}_{it} - \bar{\boldsymbol{R}}_i)'(\boldsymbol{R}_{it} - \bar{\boldsymbol{R}}_i)}{T}$$

Means

Bond	-0,4059
Stock A	3,4227
Stock B	2,7382
Stock C	1,4136

$$E[\boldsymbol{R}] = \begin{bmatrix} E[R_1] \\ E[R_2] \\ \vdots \\ E[R_N] \end{bmatrix}$$

Means and ones

-0,4059	1
3,4227	1
2,7382	1
1,4136	1

$$E[\boldsymbol{R}^*] = \begin{bmatrix} E[R_1] & 1 \\ E[R_2] & 1 \\ \vdots & 1 \\ E[R_N] & 1 \end{bmatrix}$$

Auxiliary matrix *A*

0,3708	0,1045
0,1045	0,0699

$$\boldsymbol{A} = E[\boldsymbol{R}^*]' \cdot (\sigma_{ik}) \cdot E[\boldsymbol{R}^*]$$

Target return vector

Target return	0,5
One	1

$$\boldsymbol{R}_T = \begin{bmatrix} R_T \\ 1 \end{bmatrix}$$

Portfolio weights

Bond	0,5537
Stock A	-0,1073
Stock B	0,2337
Stock C	0,3200
Sum	1

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = (\sigma_{ik})^{-1} \cdot E[\boldsymbol{R}^*] \cdot (\boldsymbol{R}_{it} - \bar{\boldsymbol{R}}_i)^{-1} \cdot \boldsymbol{R}_T$$

Portfolio properties

Variance	18,9116	$Var(R_p) = \boldsymbol{X}'(\sigma_{ik})\boldsymbol{X}$
Mean	0,5	$E[R_p] = \boldsymbol{X}'E[\boldsymbol{R}]$

Based on Figure 3.10, the following portfolio weights have been computed:

$$\boldsymbol{X} = \begin{bmatrix} Bond \\ Stock\ A \\ Stock\ B \\ Stock\ C \end{bmatrix} = \begin{bmatrix} 0.5537 \\ -0.1073 \\ 0.2337 \\ 0.3200 \end{bmatrix}$$

The portfolio mean return = 0.50, which equals the target return.

To calculate the efficient frontier it is needed to specify the target return within a suitable range and solve the Lagrange's equation again. This technique can be seen from Figure 3.11.

Figure 3.11: Computing the efficient frontier using Lagrange's method

	A	B	C	D	E	F	G	H	I	J	K	L	M
36		Auxiliary Portfolios											R_F = 0,25 %
37			Bond	Stock A	Stock B	Stock C	Variance	Exp. return	1	Std. dev.	RTVR		CML
38										0			0.25
39		Portfolio 1	0.5537	-0.1073	0.2337	0.3200	18.9116	0.5000	1	4.3487	0.0575		2.7211
40		Portfolio 2	0.4971	-0.0603	0.2732	0.2899	16.8869	0.7500	1	4.1094	0.1217		2.5851
41		Portfolio 3	0.4405	-0.0132	0.3128	0.2598	15.4445	1.0000	1	3.9299	0.1908		2.4831
42		Portfolio 4	0.3839	0.0339	0.3524	0.2298	14.5843	1.2500	1	3.8189	0.2619		2.4201
43		Portfolio 5	0.3273	0.0810	0.3920	0.1997	14.3064	1.5000	1	3.7824	0.3305	GMV	2.3993
44		Portfolio 6	0.2707	0.1281	0.4316	0.1696	14.6108	1.7500	1	3.8224	0.3924		2.4220
45		Portfolio 7	0.2141	0.1751	0.4711	0.1396	15.4975	2.0000	1	3.9367	0.4445		2.4870
46		Portfolio 8	0.1575	0.2222	0.5107	0.1095	16.9664	2.2500	1	4.1190	0.4856		2.5906
47		Portfolio 9	0.1009	0.2693	0.5503	0.0794	19.0175	2.5000	1	4.3609	0.5159		2.7280
48		Portfolio 10	0.0443	0.3164	0.5899	0.0494	21.6510	2.7500	1	4.6531	0.5373		2.8940
49		Portfolio 11	-0.0123	0.3635	0.6295	0.0193	24.8667	3.0000	1	4.9866	0.5515		3.0836
50		Portfolio 12	-0.0689	0.4106	0.6691	-0.0107	28.6646	3.2500	1	5.3539	0.5603		3.2923
51		Portfolio 13	-0.1255	0.4576	0.7086	-0.0408	33.0449	3.5000	1	5.7485	0.5654		3.5165
52		Portfolio 14	-0.1820	0.5047	0.7482	-0.0709	38.0074	3.7500	1	6.1650	0.5677		3.7532
53		Portfolio 15	-0.2386	0.5518	0.7878	-0.1009	43.5521	4.0000	1	6.5994	0.5682	TP	4.0000
54		Portfolio 16	-0.2952	0.5989	0.8274	-0.1310	49.6792	4.2500	1	7.0483	0.5675		4.2551
55		Portfolio 17	-0.3518	0.6460	0.8670	-0.1611	56.3884	4.5000	1	7.5092	0.5660		4.5170
56		Portfolio 18	-0.4084	0.6930	0.9065	-0.1911	63.6800	4.7500	1	7.9800	0.5639		4.7845
57		Portfolio 19	-0.4650	0.7401	0.9461	-0.2212	71.5538	5.0000	1	8.4589	0.5615		5.0567
58		Portfolio 20	-0.5216	0.7872	0.9857	-0.2513	80.0099	5.2500	1	8.9448	0.5590		5.3327
59		Portfolio 21	-0.5782	0.8343	1.0253	-0.2813	89.0483	5.5000	1	9.4365	0.5563		5.6122
60		Portfolio 22	-0.6348	0.8814	1.0649	-0.3114	98.6689	5.7500	1	9.9332	0.5537		5.8944
61		Portfolio 23	-0.6914	0.9284	1.1045	-0.3415	108.8718	6.0000	1	10.4342	0.5511		6.1790
62		Portfolio 24	-0.7480	0.9755	1.1440	-0.3715	119.6569	6.2500	1	10.9388	0.5485		6.4658
63		Portfolio 25	-0.8046	1.0226	1.1836	-0.4016	131.0243	6.5000	1	11.4466	0.5460		6.7543
64		***Note*** *that when inputting new data, you always need to adjust the 'Exp. Return' step size to fit the data. If the GMV portfolio or TP portfolio are indicated in the auxiliary portfolios to be in one of the extremes, this might be incorrect since the indicators are just simple Excel conditional formatting of minimum variance and maximum RTVR. Thus the correct GMV/TP portfolio might be located outside the range of the 25 auxiliary portfolios. This can be corrected by adjusting the step size in 'Exp. Return'.*											

The Capital Market Line and the RTVR is calculated using the same technique as were used in section 3.4.3.

3.6 Solver method

This section shows how Excel can compute optimal portfolio allocation when a short sales restriction is imposed. That is, short sales are disallowed. The method relies on Excel's *Solver* function (Danish: Problemløser). This method is also very easily adjusted to solve the scenario without the short sales restriction. The manual procedure described in this chapter is equivalent to the one described in Benninga (2000).

When using the Solver an error might occur, reading "Solver: An unexpected internal error or available memory was exhausted." See appendix B for a solution to the problem.

3.6.1 Solver method with manual iterations

The Excel Solver uses iterative procedures to optimize a specified optimization problem. The method relies on the Newton-Raphson procedure to find the optimal solution. The method can be considered a way of simulating the optimization to process the optimal solution. The first step is to specify the optimization problem that the Solver has to solve. As for the general case of portfolio optimization this can be written as:

Object function: $\min_{(X_1,X_2,\ldots,X_N)} \ \sigma_P^2 = \boldsymbol{X}'(\sigma_{ik})\boldsymbol{X}$ (3.33)

Constraints: $\sum X_i E(R_i) = E(R_P)$ (Portfolio return) (3.34)

$\sum X_i = 1$ (The sum of portfolio weights equals 1) (3.35)

$X_i \geq 0$ (Short sales restriction) (3.36)

$X_i \leq 1$ (no asset can exceed the weight of 1) (3.37)

It is evident that the solution to the problem must hold a minimization of the portfolio risk without violating the specified constraints. The constraints state that the sum of the portfolio weights must equal 1, and that the individual portfolio weights cannot be negative due to the fact that short sales are not allowed.

The model can be implemented in Excel as seen below in Figure 3.12.

Figure 3.12: Implementing the Solver method with short sales disallowed

Portfolio Optimization

Solver Method, Short Sales Disallowed

Variance-covariance

	Bond	Stock A	Stock B	Stock C
Bond	44.1399	5.7400	-0.8499	-1.6209
Stock A	5.7400	68.3421	4.0404	26.8383
Stock B	-0.8499	4.0404	34.0957	4.6335
Stock C	-1.6209	26.8383	4.6335	54.3028

$$(\sigma_{ik}) = \frac{(\boldsymbol{R}_{it} - \bar{\boldsymbol{R}}_i)'(\boldsymbol{R}_{it} - \bar{\boldsymbol{R}}_i)}{T}$$

Means

Bond	-0.4059
Stock A	3.4227
Stock B	2.7382
Stock C	1.4136

$$E[\boldsymbol{R}] = \begin{bmatrix} E[R_1] \\ E[R_2] \\ \vdots \\ E[R_N] \end{bmatrix}$$

Weights

Bond	0.0000
Stock A	1.0000
Stock B	0.0000
Stock C	0.0000
	1

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

Portfolio properties

Variance	68.3421	$Var(R_p) = \boldsymbol{X}'(\sigma_{ik})\boldsymbol{X}$
Sigma	8.2669	
Expected return	3.4227	$E[R_p] = \boldsymbol{X}'E[\boldsymbol{R}]$
Target return (Solver)	5.0000	
RTVR	0.3838	

It follows that the starting point is the covariance matrix and the vector of portfolio weights. This information follows from previous computations. Furthermore, a vector of portfolio weights is specified, which contains the weights found by the Solver simulation. Having determined the covariance matrix, the expected returns matrix, the portfolio weights and the target return, the variance and the expected return of the optimal portfolio can then be calculated as listed below.

Excel notation from selected cells:

G19 *=MMULT(TRANSPOSE(C19:C22);MMULT(C7:F10;C19:C22))*

G21 *=MMULT(TRANSPOSE(C19:C22);C13:C16)*

When running the Solver, inputs are set in accordance with the optimization problem specification in (3.33)-(3.36). The object function and the constraints are inputted as illustrated in the below Figure 3.13.

Figure 3.13: The solver dialog

"Target cell" refers to the object function, which in this case is an expression for minimizing the variance for a given level of target return. "By Changing Cells" refers to the portfolio weights, which are adjusted during the simulation in order to reach the optimal solution. The constraints are listed in the "Subject to the Constraints:" frame. Constraints can be added by clicking "Add".

Figure 3.14: The short sales restriction

When all the constraints have been added in dialog in Figure 3.14, click "Cancel" to return to the previous screen (Figure 3.13). When all information is set, simply press the button "Solve ". A new screen appears instantly, which reads that "Solver has found an optimal solution". Click yes to keep this Solver solution. The results are now outputted to the spreadsheet.

In this example the following results are computed, cf. Figure 3.12:

PORTOLIO WEIGHTS

Bond	0.0000 %
Stock A	0.3824 %
Stock B	0.6176 %
Stock C	0.0000 %

Portfolio return = 3.0 % (Target return)

Portfolio variance = 24.9072 %

Now, a series of portfolios must be constructed to trace out the efficient frontier, in line with the previous sections. To construct these auxiliary portfolios, the portfolio's target return must be changed in small intervals for each iteration. Subsequently, the portfolio has to be solved again after each of these changes – hence, after each iteration. For every iteration the portfolio weights, portfolio return and portfolio variance must be saved. An example of this can be seen in Figure 3.15.

Figure 3.15: Efficient frontier auxiliary portfolios

	A	B	C	D	E	F	G	H	I	J	K	L
25		Risk-free rate	0.25									
26												R_F = 0,25 %
27			Bond	Stock A	Stock B	Stock C	Mean	Variance	Std. deviation	RTVR		CML
28									0			0.25
29		Portfolio 1	0.6933	0.0000	0.0353	0.2714	0.1990	24.6946	4.9694	-0.0103		2.9882
30		Portfolio 2	0.6313	0.0000	0.1019	0.2668	0.4000	21.4068	4.6267	0.0324		2.7994
31		Portfolio 3	0.5695	0.0000	0.1683	0.2622	0.6004	18.7758	4.3331	0.0809		2.6376
32		Portfolio 4	0.5078	0.0000	0.2346	0.2576	0.8004	16.7955	4.0982	0.1343		2.5082
33		Portfolio 5	0.4463	0.0000	0.3007	0.2530	1.0000	15.4611	3.9321	0.1907		2.4167
34		Portfolio 6	0.3942	0.0232	0.3458	0.2367	1.2010	14.7073	3.8350	0.2480		2.3632
35		Portfolio 7	0.3497	0.0623	0.3763	0.2116	1.4010	14.3469	3.7877	0.3039	GMV	2.3371
36		Portfolio 8	0.3047	0.0998	0.4078	0.1877	1.6000	14.3583	3.7892	0.3563		2.3380
37		Portfolio 9	0.2600	0.1425	0.4327	0.1648	1.8000	14.7450	3.8399	0.4037		2.3659
38		Portfolio 10	0.2149	0.1789	0.4673	0.1389	2.0010	15.5035	3.9374	0.4447		2.4196
39		Portfolio 11	0.1698	0.2169	0.4980	0.1154	2.2000	16.6278	4.0777	0.4782		2.4969
40		Portfolio 12	0.1246	0.2544	0.5299	0.0911	2.4000	18.1289	4.2578	0.5050		2.5961
41		Portfolio 13	0.0791	0.2923	0.5616	0.0670	2.6010	20.0131	4.4736	0.5255		2.7151
42		Portfolio 14	0.0338	0.3299	0.5933	0.0429	2.8010	22.2614	4.7182	0.5407		2.8498
43		Portfolio 15	0.0000	0.3824	0.6176	0.0000	3.0000	24.9072	4.9907	0.5510	TP	3.0000
44		Portfolio 16	0.0000	0.6760	0.3240	0.0000	3.2010	36.5828	6.0484	0.4879		3.5828
45		Portfolio 17	0.0000	0.9682	0.0318	0.0000	3.4010	64.3514	8.0219	0.3928		4.6703
46		Portfolio 18	0.0000	1.0000	0.0000	0.0000	3.4227	68.3421	8.2669	0.3838		4.8053
47		Portfolio 19	0.0000	1.0000	0.0000	0.0000	3.4227	68.3421	8.2669	0.3838		4.8053
48		Portfolio 20	0.0000	1.0000	0.0000	0.0000	3.4227	68.3421	8.2669	0.3838		4.8053
49		Portfolio 21	0.0000	1.0000	0.0000	0.0000	3.4227	68.3421	8.2669	0.3838		4.8053
50		Portfolio 22	0.0000	1.0000	0.0000	0.0000	3.4227	68.3421	8.2669	0.3838		4.8053
51		Portfolio 23	0.0000	1.0000	0.0000	0.0000	3.4227	68.3421	8.2669	0.3838		4.8053
52		Portfolio 24	0.0000	1.0000	0.0000	0.0000	3.4227	68.3421	8.2669	0.3838		4.8053
53		Portfolio 25	0.0000	1.0000	0.0000	0.0000	3.4227	68.3421	8.2669	0.3838		4.8053

The capital market line is estimated as in Black's method from section 3.4. Based on the values computed in Figure 3.15, we can plot the CML line and the efficient frontier in a scatter plot like demonstrated in Black's method.

3.6.2 Solver method with VBA automization (for Excel 2007)

The standard method which calls for changing parameters to each iteration manually is a time-consuming task. Though, with some basic VBA programming this procedure can be turned into an efficient portfolio calculator. This extension to the Solver method can appear intimidating since some VBA programming is required but the code is provided in this section. Thus, the reader should be able to understand the most important parts of the code by studying the cell references and comparing these to the input frames in the Solver pop-up dialog box. The method in the spreadsheet works with Excel 2007 only - for an adaption to Excel 2003 see appendix A.

In brief, the VBA code is typed, edited and eventually has to be saved in a visual basic *module*, which is then linked to a button named "Solve portfolio". It is important that the solver is installed before the procedure is run.

The VBA code is presented in the following Figure 3.16. The VBA code can be seen in the visual basic editor associated with the Excel spreadsheet (click ALT+F11 to access this).

Figure 3.16: VBA code for automating the Solver optimization method

portfolio optimization incl. vba.xlsm - module1 (Code)

(General) | Doit

```
Sub Solve2()
    SolverOk SetCell:="$G$19", _
    MaxMinVal:=2, _
    ValueOf:="0", _
    ByChange:="$C$19:$C$22"
    SolverSolve UserFinish = False
    Application.ScreenUpdating = True
End Sub

Sub Doit()

jump = InputBox(Prompt:="Please state the jumps between the portfolios on the efficient frontier", _
          Title:="Inputbox", Default:=0.05)

        If jump = VBA.vbNullString Then
           MsgBox "You must state a jump interval - Restart Macro by clicking the button again"
           Exit Sub
        End If

Dim counter As Integer

    Range("C29:I53").ClearContents 'Results = C29:I53. Namelabel entered in spreadsheet.
    For counter = 1 To 25

        Range("G22") = 0 + counter * jump 'mean1 = G22
        Solve2
        Application.SendKeys ("{Enter}")
        Range("C29:I53").Cells(counter, 1) = ActiveSheet.Range("C19") 'Bond = C19
        Range("C29:I53").Cells(counter, 2) = ActiveSheet.Range("C20") 'StockA = C20
        Range("C29:I53").Cells(counter, 3) = ActiveSheet.Range("C21") 'StockB = C21
        Range("C29:I53").Cells(counter, 4) = ActiveSheet.Range("C22") 'StockC = C22
        Range("C29:I53").Cells(counter, 5) = ActiveSheet.Range("G21") 'Mean2 = G21
        Range("C29:I53").Cells(counter, 6) = ActiveSheet.Range("G19") 'Variance = G19
        Range("C29:I53").Cells(counter, 7) = ActiveSheet.Range("G20") 'Sigma = G20
        Range("C29:I53").Cells(counter, 8) = ActiveSheet.Range("G23") 'RTVR = G23

    Next counter
    Application.ScreenUpdating = True
End Sub
```

The first sub routine "Solve2()" calls the solver and is the same as the basic input dialog box for the Excel Solver function. Note the cell references to the variance and the portfolio weights. The Constraints are entered manually in Excel. If one would like to change the constraints this has to be done manually, but without pressing solve in the menu.

When pressing the solverbutton in the spreadsheet an inputbox emerges. This box asks for the specific interval change among the auxiliary portfolios. Since the solver solutions are unknown, one must enter different jumps in order to calibrate the efficient frontier. Recommended ranges vary.

The sub routine "Doit()" refers to the actual procedure for calculating various portfolios in order to estimate the efficient frontier and the capital market line. Notice different range references. An example: the range "C29:I53" refers to the output range in the spreadsheet, where the results are stored. "Counter = 1 to 25" indicates that we are simulating 25 portfolios (25 iterations). The results from each "solve" are then sent to the Results matrix. "Counter" represents each row in the matrix. By pressing the button "Solve portfolio" the VBA code is activated and the optimal portfolio is found – with automatic iterations.

For modifications of the code to include more than 1 bond and 3 stocks one would have to change the Results range to include extra columns. An example of 1 bond and 5 stocks would require the Results range to be C29:K53, and the inclusion of e.g. Stock D and Stock E.

Figure 3.17: Introducing more assets to the VBA Solver method

```
Sub Doit()

Dim counter As Integer

    Range("Results").ClearContents 'Results = C29:I53. Namelabel entered in spreadsheet.
    For counter = 1 To 25

        Range("mean1") = 0 + counter * 0.05 'mean1 = G22
        Solve2
        Application.SendKeys ("{Enter}")
        Range("Results").Cells(counter, 1) = ActiveSheet.Range("Bond") 'Bond = C19
        Range("Results").Cells(counter, 2) = ActiveSheet.Range("StockA") 'StockA = C20
        Range("Results").Cells(counter, 3) = ActiveSheet.Range("StockB") 'StockB = C21
        Range("Results").Cells(counter, 4) = ActiveSheet.Range("StockC") 'StockC = C23
        Range("Results").Cells(counter, 5) = ActiveSheet.Range("StockD") 'StockD = C24
        Range("Results").Cells(counter, 6) = ActiveSheet.Range("StockE") 'StockE = C25
        Range("Results").Cells(counter, 7) = ActiveSheet.Range("Mean2") 'Mean2 = G21
        Range("Results").Cells(counter, 8) = ActiveSheet.Range("Variance") 'Variance = G19
        Range("Results").Cells(counter, 9) = ActiveSheet.Range("Sigma") 'Sigma = G20
        Range("Results").Cells(counter, 10) = ActiveSheet.Range("RTVR") 'RTVR = G23

    Next counter
    Application.ScreenUpdating = True
End Sub
```

4 Combining Portfolio Optimization with Investor Preferences

Having determined the capital market line, all efficient portfolios have been identified as such. The market portfolio is the optimal portfolio but an investor might have different risk preferences than the market. Choosing the optimal investment for a particular investor to reflect his risk preferences is the final step of the portfolio selection process.

The optimal investment choice can be identified by introducing utility theory. In portfolio optimization expected return risk space is considered implying a trade-off between risk and return. Therefore, if the investor is only concerned about maximizing return for a given level of risk, his utility function can easily be drawn in the same expected return risk space. That is, for all levels of utility the investor's corresponding indifference curve can be derived, given a specific level of risk-aversion.

For this section the optimal solution from Black's method is applied. Consider the graphical illustration of the optimization. If the investor's utility function is quadratic, the indifference curves can be shifted upwards and to the west until the CML is exactly tangent to the indifference curve. At the tangency point, the optimal portfolio choice can be found.

Consider an investor with a quadratic utility function:

$$U(\bar{R}_P, \sigma_P) = \bar{R}_P - ½ \cdot \lambda \cdot \sigma_P^2, \tag{4.1}$$

with a risk-aversion coefficient of λ = 0.5 and:

$$\bar{R}_P = X \cdot R_X + (1 - X) \cdot R_F \tag{4.2}$$

$$\sigma_P^2 = X^2 \cdot \sigma_{TP}^2 + (1 - X)^2 \cdot 0 = X^2 \cdot \sigma_{TP}^2 \tag{4.3}$$

Inserting (4.2) and (4.3) into (4.1) yields:

$$U(\bar{R}_P, \sigma_P) = \bar{R}_P - ½ \cdot \lambda \cdot \sigma^2 \tag{4.4}$$

$$U(\bar{R}_P, \sigma_P) = X \cdot R_{TP} + (1 - X) \cdot R_F - ½ \cdot \lambda \cdot X^2 * \sigma_{TP}^2 \tag{4.5}$$

$$U(\bar{R}_P, \sigma_P) = X \cdot R_{TP} + R_f - X \cdot R_F - ½ \cdot \lambda \cdot X^2 * \sigma_{TP}^2 \tag{4.6}$$

The object is to maximize utility by adjusting the weight in the tangent portfolio *X* and the risk-free asset (*1 - X*), respectively:

$$\max_X U(\bar{R}_P, \sigma_P)$$

$$st\colon\ \bar{R}_P(\sigma_P) = R_F + \frac{R_F - R_{TP}}{\sigma_{TP}} \cdot \sigma_P = X \cdot R_{TP} + (1 - X) \cdot R_F \tag{4.7}$$

First order condition (FOC): $\frac{\partial U}{\partial X} = 0$

$$\frac{\partial U}{\partial X} = R_{TP} - R_F - 2 \cdot ½ \cdot \lambda \cdot X \cdot \sigma_{TP}^2 = 0 \tag{4.8}$$

$$R_{TP} - R_F - \lambda \cdot X \cdot \sigma_{TP}^2 = 0 \tag{4.9}$$

$$\lambda \cdot X \cdot \sigma_{TP}^2 = R_{TP} - R_F \tag{4.10}$$

$$X = \frac{R_{TP} - R_F}{\lambda \cdot \sigma_{TP}^2} \tag{4.11}$$

Thus, the optimal weight in the tangent portfolio is $X = \frac{R_{TP} - R_F}{\lambda \cdot \sigma_{TP}^2}$.

Using this result *X*, (*1 - X*), $E[R_P]$, std.dev. and *U* can easily be calculated. Below in Figure 4.1 these parameters are calculated for different values of the risk aversion coefficient λ to illustrate the convenience of the result – the optimal solution for any given level of risk aversion can be computed similarly to the λ = 0.5 case.

Figure 4.1: Market portfolio scale chosen under different levels of risk aversion

Portfolio Optimization

Combining Investor Preferences with Portfolio Optimization
- Black's Optimal Portfolio

Weights in tangent portfolio and risk-free asset chosen by particular investor preferences

Rsik aversion	Weight TP	Weight R_F	Exp. Return	Std. dev.	Utility
0.5	0.1329	0.8671	0.8808	1.1232	0.5654
1.0	0.0665	0.9335	0.5654	0.5616	0.4077
1.5	0.0443	0.9557	0.4603	0.3744	0.3551
2.0	0.0332	0.9668	0.4077	0.2808	0.3288
2.5	0.0266	0.9734	0.3762	0.2246	0.3131
3.0	0.0222	0.9778	0.3551	0.1872	0.3026
3.5	0.0190	0.9810	0.3401	0.1605	0.2951
4.0	0.0166	0.9834	0.3288	0.1404	0.2894
4.5	0.0148	0.9852	0.3201	0.1248	0.2850
5.0	0.0133	0.9867	0.3131	0.1123	0.2815

Note *that references to TP portfolio parameters in the Black's sheet are not dynamic. When inputting new data, they should be updated.*

Given the CML from Black's method, the quadratic utility function (4.1) and risk aversion level of 0.5, the maximum utility is 0.5654.

To illustrate this graphically with the efficient frontier and CML only the scenario with risk aversion level of 0.5 is considered. In order to place the utility optimization problem in mean-variance space, the utility

function must be rewritten to express return as a function of standard deviation, for a given level of risk aversion and corresponding utility. Rewriting (4.1) yields:

$$U(\bar{R}_P, \sigma_P) = \bar{R}_P - ½ \cdot \lambda \cdot \sigma_P^2$$
$$\bar{R}_P(\sigma_P) = U + ½ \cdot \lambda \cdot \sigma_P^2 \qquad (4.12)$$

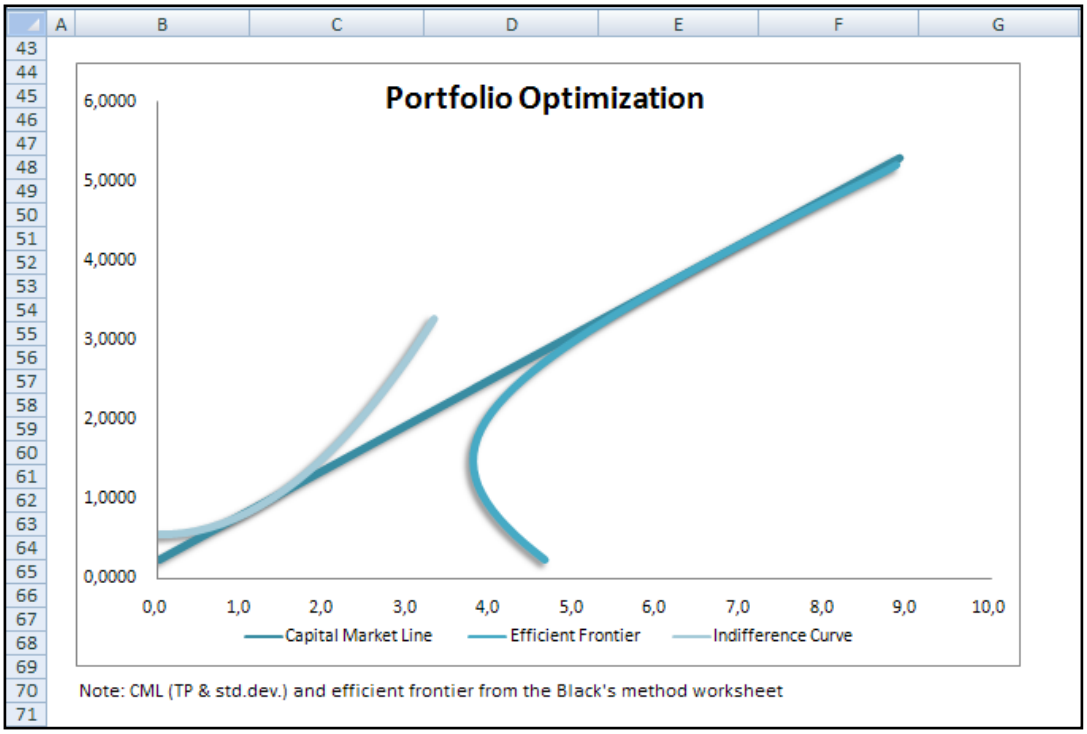
Accordingly, the following plot points can be computed to serve as coordinates for the indifference curve (Figure 4.2).

Figure 4.2: Indifference curve plot points with risk aversion = 0.5

	A	B	C	D
26				
27		Plot points for indifference curve with risk aversion = 0.5		
28		Std. dev.	Exp. return	Utility
29		0.00	0.5654	0.5654
30		0.30	0.5879	0.5654
31		0.60	0.6554	0.5654
32		0.90	0.7679	0.5654
33		1.20	0.9254	0.5654
34		1.50	1.1279	0.5654
35		1.80	1.3754	0.5654
36		2.10	1.6679	0.5654
37		2.40	2.0054	0.5654
38		2.70	2.3879	0.5654
39		3.00	2.8154	0.5654
40		3.30	3.2879	0.5654
41				

And finally, the indifference curve can be plotted with the Black's method efficient frontier and CML using the scatter plot diagramming function:

Figure 4.3: Graphical illustration of particular investor's portfolio selection



This concludes the manual with a graphical illustration of the portfolio chosen by a particular investor given his risk preferences. The optimal portfolio for this investor is found at the tangency point of CML and the indifference curve corresponding to a utility level of 0.5654.

5 Bibliography

- Articles available online are cited with a link to the article so to be accessible using your ASB login.

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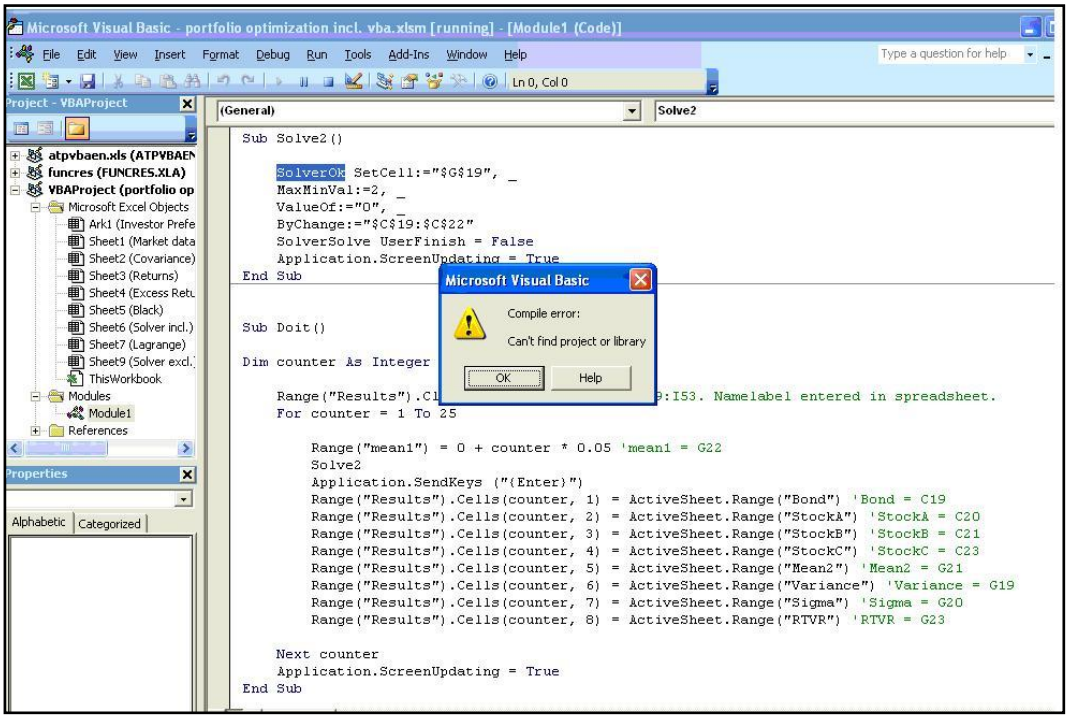
Appendix A

When the spreadsheet “Portfolio optimization incl. vba - Office 2007” is opened in Excel 2003 and one tries to run the “Solve portfolio” button/macro, an error will occur. This chapter includes a guide to correcting this error in order to enable the VBA code to work with Excel 2003 or previous versions. The steps are explained with the following screenshots. For simplicity two separate spreadsheets for Excel 2003 and 2007 are available.

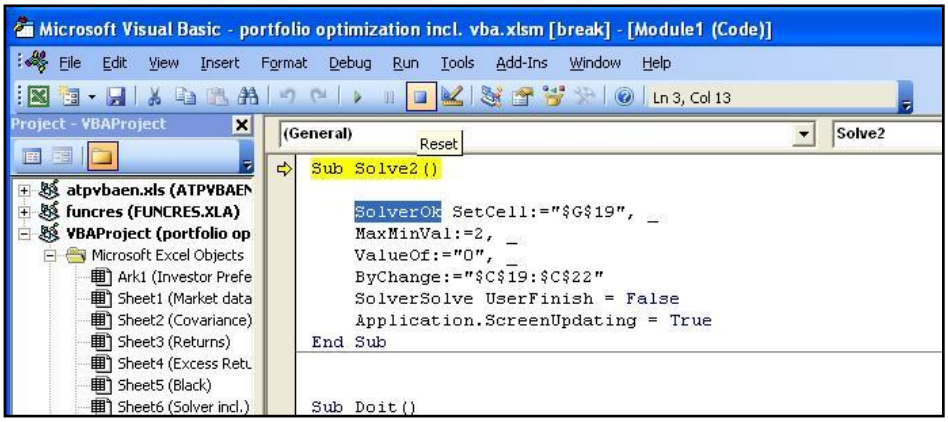
When pressing the Solver button in the spreadsheet the following error will show:

Figure A-1



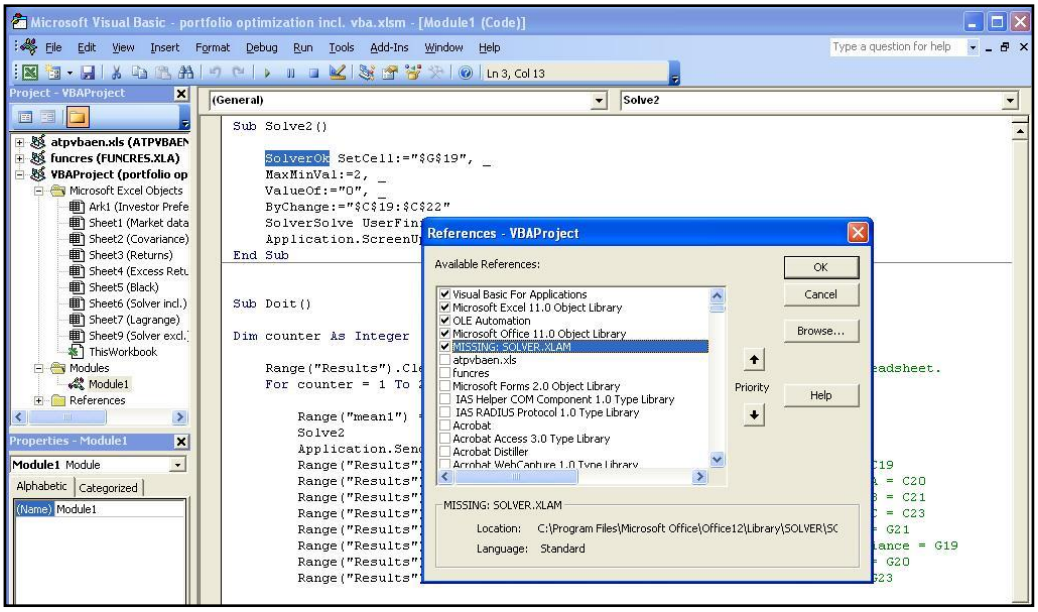
The first step is to press the square stop button in Figure A-2, to stop or reset the code. Else wise, the code cannot be edited.

Figure A-2



The error occurs due to a slight incompatibility between Excel 2003 and Excel 2007. In Excel 2007 the Solver add-in file has been moved to another location compared to Excel 2003 and therefore one has to change the reference in the VBA code to fit the correct reference when running Excel 2003. The next step is to click on Tools → References. The reference MISSING: SOLVER.XLAM should be viewable. Thus, we need to locate the Solver add-in manually and restore the reference to the correct location. Click on "Browse..." in the right hand side.

Figure A-3



The solver add-in is usually located in the following path:

*"C:\Program files\Microsoft Office**Office 11**\Library\Solver\Solver.XLA"*

The library could be Office11, Office10 or Office12 depending on the local machine. Locate the file as follows and double-click on it. Note that in the folder "*Solver*", one must choose a "*Microsoft Office Excel Files (*.xls, *.xla)*" to be able to see and select the "*Solver.XLA*"- file. See Figure A-4 and A-5 below:

Figure A-4

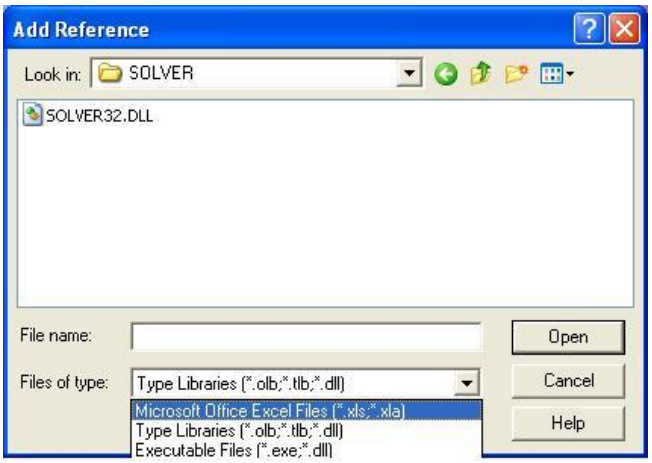
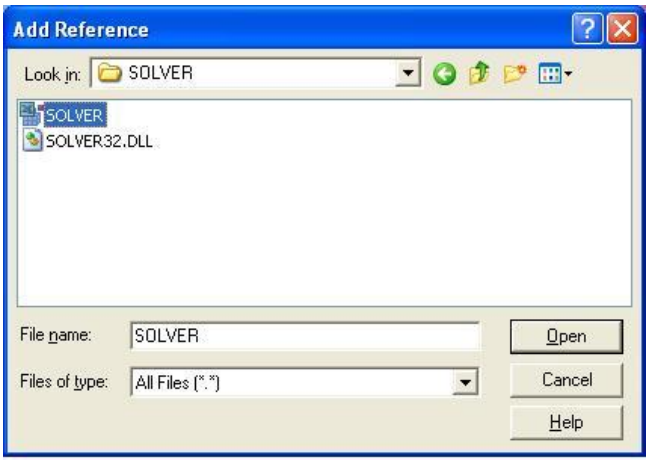
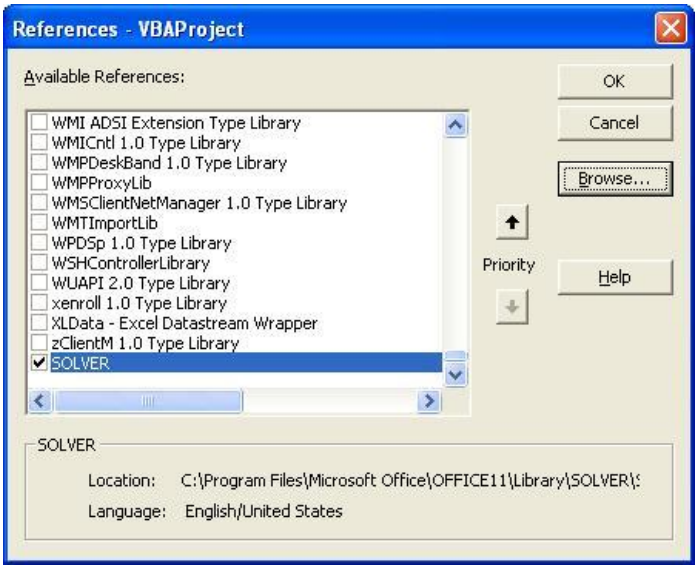


Figure A-5



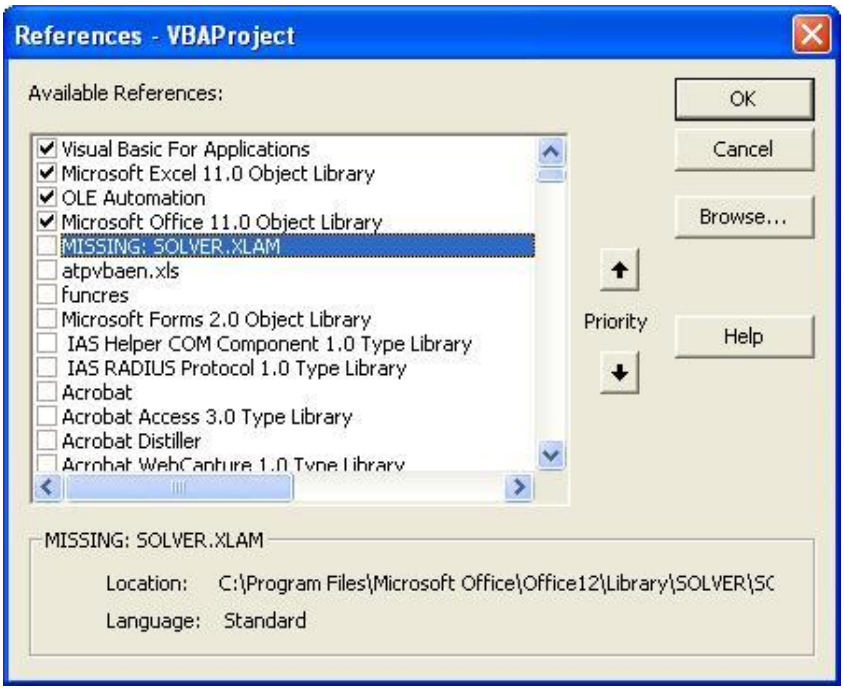
When you have located the Solver add-in and clicked on open, the following screen should be available in the References (Figure A-11):

Figure A-6



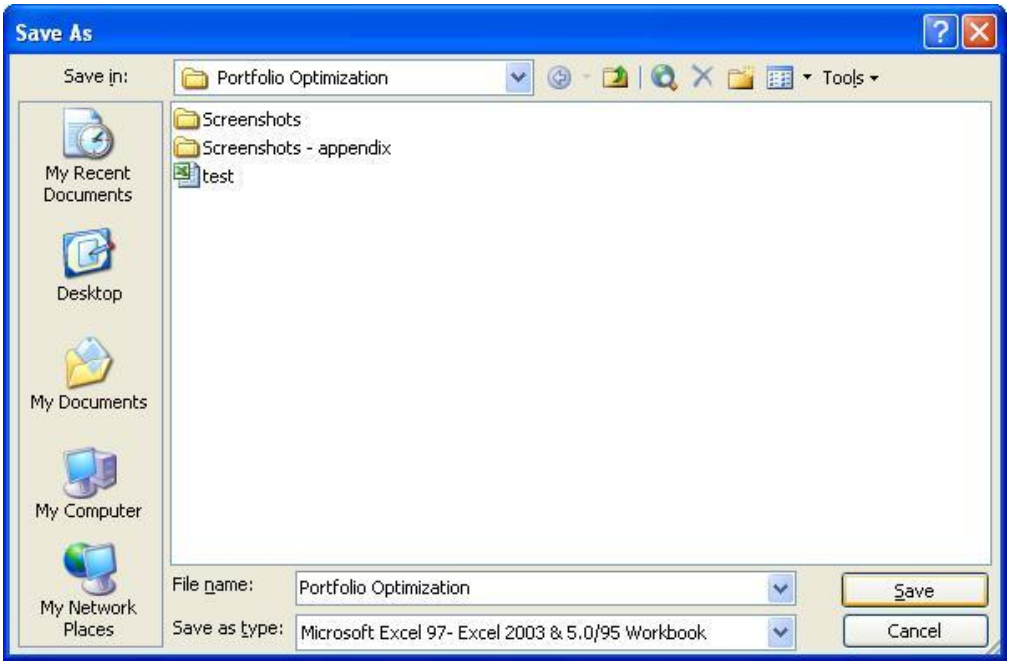
The next step is to uncheck "*MISSING: SOLVER.XLAM*" and click OK, since the correct Solver reference has now been added and the old one then must be removed. See Figure A-7 below:

Figure A-7



The final step is to save the spreadsheet as a "Microsoft Excel 97- Excel 2003 & 5.0/95 Workbook"-file (*.xls), close the spreadsheet and open it again:

Figure A-8



Now try running the Solver from the newly opened Solver worksheet again by pressing the “Solve portfolio” button. The code should now run without errors producing the optimal solution in the spreadsheet.

Appendix B

In some cases the Solver does not launch but returns you the following error message instead:
"Solver: An unexpected internal error or available memory was exhausted." (both Excel 2003 and 2007).

It is due to a flaw in the program but fortunately it can be easily bypassed. The solution is uninstall and reinstall the Solver tool add-in.

1) Therefore go to the Office button (top left) -> Click Excel option... -> Choose Add-ins in the left panel -> select Excel Ad-ins in the drop-down box -> Click Go...
2) Now uncheck the add-in Solver and click Ok.
3) Repeat 1)
4) Check the add-in Solver and click Ok.

You have now uninstalled/reinstalled the Solver and the Solver is repaired and ready to use.

Note that you must not close Excel after this procedure. In case you do, the error might pop-up again and you will have to follow the procedure above to fix the error again.